Analysis SEP 2026: Advanced Calculus (Part I)

1 Warm up: clever tricks to keep in mind!

The problems with a $\star$ next to them are ones that I actually want you to do today. The others are listed for additional practice in your own time.

1.1 Useful estimates/identities

1. Use the fundamental theorem of calculus to prove that $\sin x \leq x$ for all $x \geq 0$. Then, using this and a certain property about sine, prove that we also have that $|\sin x| \leq |x|$ for all $x \leq 0$ as well. Conclude that $|\sin x| \leq |x|, \forall x \in \mathbb{R}$. (Note that there is also a charming geometric proof of this fact!)

$\star$ 2. Prove that $1 - \cos(t) \leq t^2/2, \forall t \in \mathbb{R}$ by looking at $f(t) := 1 - \cos(t) - t^2/2$ and using Math 221 techniques to analyze what the sketch of this graph roughly looks like.

3. Prove that for $0 < \theta < 1$, $\sin \theta > \theta/2$.

1.2 Polar coordinates

$\star$ 1. For which values of $\alpha \in \mathbb{R}$ does the integral $\int_{\mathbb{R}} (1 + x^2)^{-\alpha}\, dx$ converge? **Hint:** don't over-think this, you only need Math 222 techniques to solve this.

$\star$ 2. For which values of $\alpha \in \mathbb{R}$ does the integral $\int_{\mathbb{R}^2} (1 + |x|^2)^{-\alpha}\, dx$ converge?

3. For which values of $\alpha \in \mathbb{R}$ does the integral $\int_{\mathbb{R}^n} (1 + |x|^2)^{-\alpha}\, dx$ converge?

2 Exercises

Problem 8.24.1 Let I be a compact interval. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n n^{-1/2} \cos(x/n)$$

converges uniformly on I.

Problem 8.19.2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable on $\mathbb{R}^n \setminus \{0\}$ and continuous at 0, and

$$\lim_{x \to 0} \frac{\partial f}{\partial x_i}(x) = 0$$

for $i = 1, \ldots, n$. Prove that f is differentiable at 0.

Problem 8.19.1 Let f be a continuous real valued function on $\mathbb{R}$ satisfying

$$|f(x)| \leq \frac{1}{1+x^2}.$$

Define a function F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n).$$

a) Prove that F is continuous and periodic with period 1.

b) Prove that if G is continuous and periodic with period 1, then

$$\int_0^1 F(x)G(x)\,dx = \int_{-\infty}^{\infty} f(x)G(x)\,dx.$$

Problem 8.16.3 Let $0 < \alpha < 1$. A function $f \in C([0,1])$ is said to be Hölder continuous of order α if there is a constnat $C > 0$ such that

$$|f(x) - f(y)| \leq C|x-y|^{\alpha}$$

for all $x, y \in [0,1]$. Show that for every $0 < \alpha < 1$, the function

$$f(x) = \sum_{n=1}^{\infty} 2^{-n\alpha} \cos(2^n x)$$

is Hölder continuous of order α.

Problem 1.07.3 Let I be a compact subset of $(0, 2\pi)$. Show that the series

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

converges uniformly on I.

3 Homework :-)

The "basic analysis/advanced calculus" section of the qual is typically questions 1–3 on the exam. Try to go through as many as you can on your own. Make note of the ones that you do not have any ideas on how to solve (i.e., without hints from your peers or the internet).